Log. Univers. 5 (2011), 127–164

1661-8297/11/010127-38, *published online* March 9, 2011
DOI 10.1007/s11787-011-0026-5

Logica Universalis

The Method of Scientific Discovery in Peirce's Philosophy: Deduction, Induction, and Abduction

Cassiano Terra Rodrigues

Abstract. In this paper we will show Peirce's distinction between deduction, induction and abduction. The aim of the paper is to show how Peirce changed his views on the subject, from an understanding of deduction, induction and hypotheses as types of reasoning to understanding them as stages of inquiry very tightly connected. In order to get a better understanding of Peirce's originality on this, we show Peirce's distinctions between qualitative and quantitative induction and between theorematical and corollarial deduction, passing then to the distinction between mathematics and logic. In the end, we propose a sketch of a comparison between Peirce and Whitehead concerning the two thinkers' view of mathematics, hoping that this could point to further inquiries.

Mathematics Subject Classification (2010). Primary 03A10; Secondary 00A30.

Keywords. Deduction, induction, abduction, hypothesis, logic of discovery, theorematical and corollarial deduction, nature of mathematics, Peirce, Whitehead.

In natural science this rigid method is the Baconian method of induction, a method which, if consistently pursued, would have left science where it found it. What Bacon omitted was the play of free imagination, controlled by the requirements of coherence and logic. The true method of discovery is like the flight of an aeroplane. It starts from the ground of particular observation; it makes a flight in the thin air of imaginative generalization; and it again lands for renewed observation rendered acute by rational interpretation.
Alfred North Whitehead, *Process and Reality.*

Birkhäuser

1. Deduction

In 1868, Peirce distinguished kinds of valid inferences in syllogistic terms: apodictic (or deductive) syllogisms, and probable (or inductive and hypothetical) syllogisms. “An apodictic or deductive syllogism is one whose validity depends unconditionally upon the relation of the fact inferred to the facts posited in the premisses” [30, 2: 215: *Some Consequences of Four Incapacities*]. That the validity of deduction does not depend upon the experience of an ulterior knowledge, but only upon its premisses, is a peculiar characteristic of deduction: either this other knowledge would be in the premisses, and obviously would not be another, or it would be implicit, and the inference would be incomplete. For this reason, deduction is the only kind of reasoning that can be called necessary [25, p. 217]. The truth of deduction is warranted in a necessary manner in that all the possible relevant information needed to reach the conclusion is in the premisses [30, 2: 175: *Questions on Reality*, 1868]. Deduction, thus, can be showed to have the following syllogistic form in *Barbara*:

Rule:	All beans in this bag are white	All S is P
Case:	These beans are from this bag	M is S
Result:	These beans are white	M is P

It is an important conclusion that in all necessarily valid deductive reasoning from true premisses, it is possible to draw only true conclusions [30, 2: 175]. As a matter of fact, this position will always be maintained by Peirce in his writings; see the following passage, from 1903:

In deduction, or necessary reasoning, we set out from a hypothetical state of things which we define in certain abstracted respects. Among the characters to which we pay no attention in this mode of argument is whether or not the hypothesis of our premisses conforms more or less to the state of things in the outward world. [...] Our inference is valid if and only if there really is such a relation between the state of things supposed in the premisses and the state of things stated in the conclusion [25, p. 225].

The principle stated since 1868 remains the same, which says that true premisses yield true conclusions. Now, the agreement with the concrete facts, so to say, is another problem, that does not affect the validity of the deduction. All deduction is reasoning starting from the general to reach the particular, meaning that it is an inference according to a general rule of drawing particular propositions from general propositions. For that reason, the necessity of the reasoning is in that the truth of conclusions depends on the truth of premisses [27, 3/I: 172: *Probability and Induction*, 1911]. In agreement with this idea, deduction is also denominated by Peirce analytical inference, because the result attained cannot be discordant from the rule [30, 3: 323–326: *Deduction, Induction, and Hypothesis*, 1878].

Notwithstanding, Peirce further recognized that deductively reached conclusions may not be absolutely valid. Deductions are also forms of probable reasoning, even when the probability of extracting false conclusions from true premisses is minimal; that is, deductions are inferences that "in the long run of experience the greater part of those whose premisses are true will have true conclusions." [24, 2: 298, *Nomenclature and Divisions of Triadic Relations, as Far as They Are Determined*, 1903]. In this respect, it may be more correct to say that there are necessary deductions and probable deductions, meaning that these deductions are of a certain probability [27, 3/I: 172, *Probability and Induction*, 1911]. This is a very strong consequence of Peirce's well-known fallibilism.[1]

The first important point we want to emphasize is that the logical validity of reasoning depends essentially on the obeisance of the rule for passing from premisses to conclusion:

The passage from the premiss (or set of premisses) P to the conclusion C takes place according to a habit or rule active within us. [...] The habit is logically good provided it would never (or in the case of a probable inference, seldom) lead from a true premiss to a false conclusion; otherwise it is logically bad. [30, 4: 165, *On the Algebra of Logic*, 1880].

In other words, a good habit of reasoning takes into account a disjunction of the following form: or the premiss is false, or the conclusion is true. A bad habit of reasoning extracts false conclusions from true premisses [29, p. 167]. This is important, for Peirce is not questioning the very necessity of reasoning, but only the correctness of the procedure: it is possible that the reasoner blunders on any step, but the apodicticity of deduction is not affected by such blunders [21, pp. 230–231]. So, any mistake can occur in a deduction, and false conclusions can be obtained starting from true premisses, as just mentioned, resulting from a bad deductive habit or some inattention in reasoning. This inattention is a matter of observation, and the conclusion is that observation and necessity are not definitely severed one from another; rather, they are rather two different steps in reasoning:

Deduction is really a matter of perception and of experimentation, just as induction and hypothetic inference are; only, the perception and experimentation are concerned with imaginary objects instead of with real ones. The operations of perception and of experimentation are subject to error, and therefore it is only in a Pickwickian sense that mathematical reasoning can be said to be perfectly certain. It is so only under the condition that no error creeps into it; yet, after all, it is susceptible of attaining a practical certainty.[2] [23, 6.595, *A Reply to Necessitarians*, 1893].

So, the second important point to highlight follows from this and is that, *theoretically*, error is impossible in deduction, there is nothing wrong with apodicticity, but there might be something wrong with the observation: "The truths of mathematics are truths about ideas merely. They are all but certain. Only blunderings can introduce error into mathematics." [28, p. 37: *The Essence of Reasoning*, 1893]. Now, the link between necessary reasoning and

[1] Peirce's ontological fallibilism cannot be dealt with in full detail here. For more on the subject, see the excellent article [34].

[2] Peirce alludes here to Charles Dickens' first novel, The Posthumous Papers of the Pickwick Club.

mathematical reasoning lies in the fact that the mathematician proceeds in a definite way from observation of imaginary constructions. This is an important point and we will return to it later, when discussing the differences between mathematics and logics.

There are, then, two cases of possible error: first, since deductions concern mathematical reasonings on imaginary items, necessity comes from the strict obeisance of the rule of transitivity from premisses to conclusions, but some error might occur in observation; and second, the possibility of application of the conclusions leaves open the possibility of an error, for probability is at stake then:

Deduction is the only necessary reasoning. It is the reasoning of mathematics. It starts from a hypothesis, the truth or falsity of which has nothing to do with the reasoning; and of course its conclusions are equally ideal. [...] Moreover, its application to experience, or to possible experience, opens the door to probability, and shuts out absolute necessity and certainty, *in toto.* [23, 6.595: *A Reply to Necessitarians*, 1893].

So, the important aspect of the typical logical form of deduction is understood by Peirce as one of a syllogism, the major term of which must be a universal categorical proposition, and the minor term of which an affirmative proposition, seen in the syllogism of the first figure (in *Barbara*), as previously quoted. And until 1878, to pass to the other kinds of reasonings—induction and hypothesis—it is important to keep this syllogistic form in mind. Deduction appears as the only kind of valid reasoning which is necessary; induction and hypothesis, though not necessary, are also valid, as we shall see in a minute. Before turning to induction and hypothesis, it is necessary to say that Peirce maintained his position concerning deduction throughout his life. He also came to distinguish between theorematical and corollarial deduction, a distinction upon which we will turn soon. But, in order to grasp this distinction correctly, we need to remember another point Peirce never dismissed: deduction, induction and hypothesis are the only kinds of valid reasoning, and all thought is of some one of these kinds, or a combination of them, and all thought is of some one of these kinds, or a combination of them. This is a conclusion Peirce holds since at least since 1868 [30, 2, *passim*], and that opens the way for concluding that there is no difference in nature between practical and theoretical reasoning.[3] The basal difference he in 1868 established between the three kinds of reasoning is that deduction is a demonstrative inference that does not increase our knowledge of facts, for it passes from a general rule, as a premise, to affirm a conclusion resulting from the analytical unfolding of the relations affirmed in the premise; induction and hypotheses, differently, are ampliative illations, different one from another, that increase our knowledge: induction is a generalization, it proceeds from the verification of a particular experience to infer a general explanatory rule for such experience; hypothesis, in its turn, assumes a general rule from the beginning, and from it seeks to

[3] This is a point Apel never gets tired of stressing, for it's one of Peirce's main steps in his overcoming of Kantian transcendentalism. See [1].

establish relations between particular facts of experience, otherwise apparently disconnected. Let us now pass to the details.

2. Induction and Hypothesis as Transformations of Deduction

A typical inductive reasoning would have the inverted form of a deductive syllogism as follows:

Case:	These beans are from this bag	M is S
Result:	These beans are White	M is P
∴ Rule:	∴ All the beans in this bag are white	∴ All S is P

It is possible to understand induction as a form of inversion of the deductive inference, starting from the second premiss as its major one to the conclusion of the rule; or, in other words, a *probable deduction* consisting in the inference of a general rule from the observation of a result in a certain case [30, 3: 328, *Deduction, Induction, and Hypothesis*, 1878]. Then, deriving the major premiss of a syllogism from its minor premisses, induction is a form of reduction of the multiplicity to the unity, allowing for an assertion about facts, very likely to be true [30, 3: 217]. Induction, amplifying the extension of a certain class of subjects, amplifies the generality of the conclusion beyond the limits affirmed in the major premiss of the syllogism, in an operation that allows to pass from the determination of the existence to the virtuality of the possible, for it reaches a general concept about *actual* instances; for such reason we can assure that other similar instances can be submitted to the same concept.

Let us see hypothesis:

Hypothesis can be defined as an argument which proceeds upon the assumption that a character which is known necessarily to involve a certain number of others, may be probably predicated of any objects which have all the characters which this character is known to involve. [30, 2: 217–218].

According to such definition, hypothesis can be defined as the inference of a minor premiss from two other premisses of a syllogism; in other words, as an inference of a particular instance from a general rule and the probable result of the application of the rule to the case, in another kind of the inversion of deduction [30, 3: 325–328].

Hypothesis can be written as follows:

Rule:	All the beans of this bag are white	All S is P
Result:	These beans are white	M is P
∴ Case:	∴ These beans are from this bag	∴ M is S

We then see that through the inversion of the deductive syllogism, changing the places of subjects and predicates, that is, changing premisses, we get different kinds of reasoning. Peirce can conclude then, in 1883:

Deduction proceeds from Rule and Case to Result; it is the formula of Volition. Induction proceeds from Case and Result to Rule; it is the formula of the formation of a habit or general conception—a process which, psychologically as well as logically, depends on the repetition of instances or sensations. Hypothesis proceeds from Rule and Result to Case; it is the formula of the acquirement of secondary sensation—a process by which a confused concatenation of predicates is brought into order under a synthesizing predicate. [30, 4: 422, *A Theory of Probable Inference*].

Later on, however, Peirce came to modify his conceptions in specific points. The first significant change is that deduction, induction and *abduction,* which is the kind of reasoning to formulate hypotheses, are understood as stages of scientific inquiry. Another very important change, the more important one for our present purposes, concerns the heuretic status of induction: induction is not anymore considered as a knowledge-ampliative form of inference; only *abduction* has the power to amplify knowledge, for its meaning is to formulate hypotheses—induction is considered just as the experimental testing of devised hypotheses. As these definitions and relations are very complex and intricate, we can here only sketch an introductory account.

3. The Stages of Scientific Inquiry

First, let us define probability. Probability is defined as the ration of frequency between the occurrence of known facts and the occurrence of unknown facts:

The *probability* that if an antecedent condition is satisfied, a consequent kind of event will take place is the quotient of the number of occasions, 'in the long run', in which both the antecedent will be satisfied and the consequent kind of event will take place, divided by the total number of occasions on which the antecedent conditions will be satisfied. [27, 3/I: 174].

The idea of the long run is essential because it links the ideas of probability and convergence: the method of induction gives us the possibility of asserting that, in the long run, the frequency of a certain kind of event *would tend* to a definite value, which is not yet known; in other words, the ratio between the number of times in which this event could occur and the number of times in which the proper occasion for this happening arises would indefinitely converge towards a limit. The idea of convergence is explained as follows:

The word "converge" is here used in a different sense from that which is usual in mathematics. The common definition is that a series of values, x_1, x_2, x_3, etc., converges toward a limiting value x, provided, after any discrepancy ε has been named it is possible to find one of the members of the series x_v such that, for every value of n greater than v, $(x_n - x) < \varepsilon$. *This ought to be called definite* convergence. No such member, x_n can in the indefinite convergence with which we have to do, be fixed in advance of the experiment. Nevertheless, there will be some such value. [26, 2: 745, *On the Logic of Drawing History from Ancient Documents, Especially from Testimonies*, 1901].

Now, it is possible to discover a ratio, that is, an approximate proportion of occurrence, a statistical probability: "Objective probability is simply a statistical ratio." [27, 4: 59, *Carnegie Application*, 1902]. It means that our knowledge is reduced to a merely probable estimation: scientific discoveries are but attempts to diminish our errors. In other words, from an initial sample, to endeavour to define the probable characters of the whole universe, correcting the deviations through successive qualitative inductions.[4] In sum, any and every scientific assertion can be but probable:

To say, for instance, that the demonstration by Archimedes of the property of the lever would fall to the ground if men were endowed with free will is extravagant; yet this is implied by those who make a proposition incompatible with the freedom of the will the postulate of all inference. Considering, too, that the conclusions of science make no pretense to being more than probable, and considering that a probable inference can at most only suppose something to be most frequently, or otherwise approximately, true, but never that anything is precisely true without exception throughout the universe, we see how far this proposition in truth is from being so postulated. [24, 1: 299, *The Doctrine of Necessity Examined*, 1892].

Scientific knowledge, therefore, is a way of understanding the distribution of the characters in the sample with relation to the whole supposed universe. The idea of pre-designation is important in that scientific knowledge is grounded in the recognition of *how* the sample presents certain qualities. Suppose that a ship be loaded with wheat, as Peirce says [*id.*]. This load is stirred up so that the grains are all mixed up. Samples are equally taken out from the fore, amidships, and from the aft parts, from larboard as well as from starboard, from the top, half depth and bottom of her hold, and, after analyzing them: "Then we infer, experientially and provisionally, that the approximately four-fifths of all the grain in the cargo is of the same quality." [24, 1: 301]. The estimated frequency has nothing to do with some wheat that might have possibly been hidden in the ship and is not drawn in the sample:

By saying that we infer it *experientially*, I mean that our conclusion makes no pretension to knowledge of wheat-in-itself. [...] We are dealing only with the matter of possible experience – experience in the full acceptation of the term as something not merely affecting the senses but also as the subject of thought. [...] By saying that we draw the inference *provisionally*, I mean that we do not hold that we have reached any assigned degree of approximation as yet, but only hold that if our experience be indefinitely extended, and if every fact of whatever nature, as fast as it presents itself, be duly applied, according to the inductive method, in correcting the inferred ratio, then our approximation will become indefinitely close in the long run; that is to say, close to the experience *to come* (not merely close by the exhaustion of a finite collection) so that if experience in general is to fluctuate irregularly to and fro, in a manner to deprive the ratio sought of all definite value, we shall be able to find out approximately within what limits it fluctuates, and if, after having one definite value, it changes and assumes another, we shall be able to find that out, and in short, whatever may be the variations of this ratio in experience,

[4] In fact, no scientific affirmation can be any more than probable. Peirce vehemently rejects any absolute and ever-fixed necessity, certainty and truth whatsoever.

experience indefinitely extended will enable us to detect them, so as to predict rightly, at last, what its ultimate value may be, if it have any ultimate value, or what the ultimate law of succession of values may be, if there be any such ultimate law, or that it ultimately fluctuates irregularly within certain limits, if it do so ultimately fluctuate. [24, 1: 301].

We have seen that induction is the kind of inference that allows for the passage of the particular to the general. In effect, relatively to this respect, there is the idea of pre-designating the character to be discovered. Samples are drawn at random, to be examined as to some specific respect; in the case, of which quality is the wheat. Induction makes possible to ascertain a general character to the whole cargo based upon the fact that the samples are of a certain quality. Through successive sampling a conclusion is reached, that most likely defines the whole. The procedure can be repeated as many times as wanted. And in fact, the more it is repeated, the greater assurance the generalization will attain. Thus, if there is some truth to be discovered, it can be affirmed in terms of the general character, which can be attributed to a whole class or infinite series, from the verification that a character of the same species in its members to which we had access; in other words, induction is a reasoning which allows to recognize what is true of the whole, in recognizing a general true character of the parts [27, 3/I: 182, letter to Kehler, 1911].

The idea of induction as synecdoche is from early on present in Peirce's writings. Nevertheless, he came to distinguish three different kinds of induction, to wit, the *crude* or *rudimentary* induction, *qualitative* induction, and *quantitative* induction [23, 7.110–130, *Lowell Lectures*, 1903]. Each one of these sorts of induction relates in a different way the particular to the general, and it is worthwhile to analyse them more thoroughly. It is with the idea of quantitative induction that Peirce recovers Fermat's inference, affirming to be possible to ascertain in a quantitative way certain distinctive qualities, so to allow the identification of the events in classes.

3.1. Types of Induction

The first kind of induction is defined in the following way: "By 'crude' induction, I mean that inarticulate, unreflective kind usually but very inappropriately termed (I suppose in imitation of Francis Bacon) *inductio per simplicem enumerationem* [*induction by simple enumeration*]." [27, 3/I: 214, *On the Foundation of Ampliative Reasoning*, 1910]. In this kind of rudimentary induction "the collection to be sampled is an objective series of which some members have been experienced, while the rest remain to be experienced, and we simply conclude that future experience will be like the past." [26, 2: 748].

This is the weakest kind of induction [26, 2: 749], for it is based on the absence of knowledge, i.e.: "*Rudimentary induction* [...] proceeds from the premiss that the reasoner has no evidence of the existence of any fact of a given description and concludes that there never was, is not, and never will be any such thing." [23, 7.111]. It is a self-correcting method, since the experimental series is not interrupted; "and if the series of observations skips a single day, that day may be the very day of the exceptional fact." [*id.*]. In other words, the only conclusion crude induction warrants is that there is not *yet* enough evidenceto abandon the initial hypothesis. Its weakness is that if contrary evidence to the initial assumption is discovered, then it needs to be dismissed as invalid. As it is not a question of probabilities, there is no reason why to assume the falsity of the conclusion; but there is not also a higher reason to assume its truth *in the future*. In other words, the probability of the conclusion being true is 50%. This is a method which, relative to "gratuit hypothesis" it furnishes some degree of assurance; it is impossible not to use it some time. Even then, it is the weakest kind of induction. [*ibid*.].

Another kind of induction is qualitative induction: "This kind of reasoning may be described [. . .] by saying that it tests a hypothesis by sampling the possible predictions that may be based upon it." In more detailed terms, the process consists in the following steps:

I seem to recognize a [. . .] genus of inductions where we draw a sample of an aggregate which cannot be considered as a collection, since it does not consist of units capable of being either counted or measured, however roughly; and where probability therefore cannot enter; but where we can draw the distinction of much and little, so that we can conceive of measurement being established; and where we may expect that any error into which the sampling will lead us, though it may not be corrected by a mere enlargement of the sample, or even by drawing other similar samples, yet must be brought to light, and that gradually, by persistence in the same general method. [26, 2: 750–751].

In general lines, then the process above described is equivalent to the hypothetical-deductive method of verification of theories. Phenomena are observed, apparently disconnected, that is, apparently they do not make up the same collection. Two possibilities are then open to the inquirer: "In the first place, we may look through the known facts and scrutinize them carefully to see how far they agree with the hypothesis and how far they call for modifications of it." [23, 7.114]. In other words, we seek to understand new facts creating general conceptions based upon what we already know. This is the process of *abduction*, that is, of creation of explanatory hypothesis for the facts, and it is formally very much like induction, for it also starts from the particular to reach the general. Nevertheless, to take one for the other is to commit a fallacy *post hoc, ergo propter hoc* (literally: *after that, then because of that*), i.e., to confound the cause with what is not the cause (yet: to affirm the consequent). For instance, to think that a fact A, when temporally previous to B, only for being previous, is the cause of B. But A can be previous to B without being its cause, in the same way that B may come after A without being its consequence.[5] If this fallacy not committed, the procedure of looking for the facts we know in the attempt to understand those we do not know is of a great value to inquiry [*id*.].

[5] Aristotle gives an instance of this fallacy in Sophistical Refutations, 167 b21–167 b36. See [2].

The second attempt is not to look at the facts, but to the hypothesis that we have and to test their capacity to predict what will happen in the future:

The other line which our studies of the relation of the hypothesis to experience may pursue, consists in directing our attention, not primarily to the facts, but primarily to the hypothesis, and in studying out what effect that hypothesis, if embraced, must have in modifying our expectations in regard to future experience. [23, 7.115].

In such way bounded by empirical check, the hypotheses that better describe the run of experience are continually tested, until the expectations they created be contradicted by facts. Now, this is the first step of scientific investigation, for it starts from a surprise in experience to get to the hypothesis that explains it [23, 2.755, c. 1905]. This reasoning Peirce called *retroduction*:

The whole series of mental performances between the notice of the wonderful phenomenon and the acceptance of the hypothesis, during which the usually docile understanding seems to hold the bit between its teeth and to have us at its mercy, the search for pertinent circumstances and the laying hold of them, sometimes without our cognizance, the scrutiny of them, the dark laboring, the bursting out of the startling conjecture, the remarking of its smooth fitting to the anomaly, as it is turned back and forth like a key in a lock, and the final estimation of its Plausibility,—I reckon as composing the First Stage of Inquiry. Its characteristic formula of reasoning I term Retroduction, i.e. reasoning from consequent to antecedent. [24, 2: 441, *A Neglected Argument for the Reality of God*, 1908].

The reasoning is called retroduction exactly because the framing of the hypothesis begins with the observation of a striking fact. Its logical form is the following:

The surprising fact, C, is observed;

But if A were true, C would be a matter of course.

Hence, there is reason to suspect that A is true. [25, p. 245].

We can indeed understand the form of a deductive fallacy here, namely, the fallacy of affirming the consequent.[6] But more important is to notice that inquiry begins with the observation of something that appears as striking, something not agreeable to what is expected, interrupting an habit of expectation. The business of inquiry is to inquiry into those phenomena, to devise an explanatory hypothesis to give an account of such "wondering". Thus, the

[6] This is a widespread understanding of abduction nowadays, and it usually appears in discussions about the similarities and differences between abduction and the inference for the best explanation. But this approach is not untroubled. We cannot deal with this theme here, but the reader will find some discussion of it in [5] and [6]. But let us remember that Peirce's theory is broader in scope than a methodological theory of scientific explanations. Suffice by now to quote the following: "I ought to say that when I described induction as the experimental testing of a hypothesis, I was not thinking of experimentation in the narrow sense in which it is confined to cases in which we desire to study a phenomenon. I mean to extend it to every case in which having ascertained by deduction that a theory would lead us to anticipate under certain circumstances phenomena contrary to what we should expect if the theory were not true, we examine the cases of that sort to see how far those predictions are borne out." [24, 2: 234].

very first step towards discovering truth is in the imagination of what this truth could be:

As soon as a man experiences a longing to know the truth, he begins to imagine what that truth can be. He very soon finds that unrestrained imagination is sure to lead him wrong. Nevertheless, it remains true that imagination alone, - under proper checks, - can possibly suggest the truth. Hence, the second requisite for the successful pursuit of science, - coming out after the desire to learn, - is a scientific and fertile imagination. [...] The scientific man dreams of agencies by which the phenomena of nature might be brought about. [26, 2: 1117–1118, *The Chief Lessons of the History of Science*, 1892].

According to this, and in the same vein of Whitehead's quote taken as epigraph for this article, Peirce states:

Tennyson says:

maybe wildest dreams

are but the needful preludes of truth.

But I would dock the *maybe*. Wildest dreams *are* the necessary "first steps toward scientific investigation". [26, 1: 157, *Early History of Science*, 1892].

The imagination of hypotheses is the essential first step of science, in its search for truth. In fact, the retroductive process of imagining hypotheses is the only one which is endowed with original heuristic power:

Abduction is the process of forming an explanatory hypothesis. It is the only logical operation which introduces any new idea; for induction does nothing but determine a value, and deduction merely evolves the necessary consequences of a pure hypothesis.

Deduction proves that something *must* be; Induction shows that something *actually is* operative; Abduction merely suggests that something *may be*. [25, p. 230].

The *modus operandi* of scientific method is that from imagined hypothesis it is retroductively possible to reach certain conclusions necessarily. Notice that abduction itself does not put any necessity on the hypotheses it suggests, for they can be used only in a deductive reasoning in the place of the *premiss*. The conclusion necessarily obtained from this surmise will be inductively tested in experience, in order to be possible to discard the conclusions which do not describe facts properly:

Deduction is the only necessary reasoning. It is the reasoning of mathematics. It starts from a hypothesis, the truth or falsity of which has nothing to do with the reasoning; and of course its conclusions are equally ideal. [...] Induction is the experimental testing of a theory. The justification of it is that, although the conclusion at any stage of the investigation may be more or less erroneous, yet the further application of the same method must correct the error. The only thing that induction accomplishes is to determine the value of a quantity. It sets out with a theory and it measures the degree of concordance of that theory with fact. It never can originate any idea whatever. No more can deduction. All the ideas of science come to it by the way of Abduction. Abduction consists in studying facts and devising a theory to explain them.

Its only justification is that if we are ever to understand things at all, it must be in that way. [25, pp. 217–218].

Now, a significant change relative to his early writings is that Peirce abandons the attempt to explain deduction as a syllogism in Barbara. Saying that it starts with an hypothesis is another way of saying that it establishes the necessary from a conditional; it is not restricted, then, to passing from the universal to the particular. Now, the function of the hypothesis as the major premiss is decisive, since the only justification for the validity of abduction is its capacity to open new boundaries [27, 3/I: 206]. So, it needs to be tested in experience. Abductions are like Whitehead's play of free imagination, which must be made acute by the coherence with facts and by logical consistency. The process of imagining hypotheses and their following test are characteristic of qualitative induction. Qualitative induction thus is the "mix" [32, p. 3][7] of two processes: the abductive, or retroductive, the imagining of a hypothesis, and the inductive, the testing of it. After successive eliminations of explanatory hypothesis through empirical testing, an explanation will be reached to exhibit the striking fact as the conclusion of a deductive syllogism. This explanatory hypothesis finally may be account as plausible [24, 2: 441]. In other words: "This kind of reasoning may be described in slightly different terms by saying that it tests a hypothesis by sampling the possible predictions that may be based upon it." [26, 2: 751]. The development and working out of this method is tied-linked to the collective practice of science:

One generation collects premisses in order that a distant generation may discover what they mean. When a problem comes before the scientific world, a hundred men immediately set all their energies to work upon it. One contributes this, another that. Another company, standing upon the shoulders of the first, strike a little higher, until at last the parapet is attained. Still another moral factor of the method of science, perhaps even more vital than the last, is the self-confidence of it. In order to appreciate this, it is to be remembered that the entire fabric of science has to be built up out of surmises at truth. All that experiment can do is to tell us when we have surmised wrong. The right surmise is left for us to produce. [23, 7.87, *Scientific Method*, 1902].

Again, the point about induction is that it consists only in the experimental testing of a hypothesis; only abduction has the capacity to suggest

[7] In fact, Rescher distinguishes abduction from retroduction. In this regard, we follow the account provided Parker, in [22] p. 175, in a criticism to Rescher. Now, one notable exception to what was said in the previous note 6 was Hanson [10], who, in his Patterns of Discovery, recovered Peirce's approach to Kepler's discovery process, interpreting it in terms of abduction. This is a discussion we cannot pursue within the limits of this paper, but it is worth of mentioning at least that Hanson, in contrast to Rescher, and following Peirce, defines abduction in terms of retroduction: "Theories put phenomena into systems. They are built up 'in reverse'—retroductively. A theory is a cluster of conclusions in search of a premiss. From the observed properties of phenomena the physicist reasons his way towards a keystone idea from which the properties are explicable as a matter of course." This is taken from [10], p. 90. A full account of Hanson's originality on the subject as well as his indebtedness to Peirce's ideas is yet to be written. We cannot but allude to this here.

new discoveries. This will become clearer if we consider quantitative induction. Take one of its particularly clear definitions:

This [form of induction] investigates the interrogative suggestion of retroduction, 'What is the 'real probability' that an individual member of a certain experiential class, say the S's, will have a certain character, say that of being P?" This it does by first collecting, on scientific principles, a "fair sample" of the S's, taking due account, in doing so, of the intention of using its proportion of members that possess the predesignate character of being P. This sample will contain none of those S's on which the retroduction was founded. The induction then presumes that the value of the proportion, among the S's of the sample, of those that are P, probably approximates, within a certain limit of approximation, to the value of the real probability in question. I propose to term such reasoning *Quantitative Induction*. [23, 2.758, c. 1905].

Quantitative induction seeks to ascertain a quantity, and nothing more; in other words, it measures the degree of concordance of the theory with the facts. For such reason, its success is relative to the amount of extra information not contained in the premisses which it is possible to gather: the probability of its conclusions being true is proportional to the quantity of positive evidence which is gathered to prove a theory, "for induction proper consists in judging of the relative frequency of a character among all the individuals of a class by the relative frequency of that character among the individuals of a random sample of that class." [23, 6.100, *Uniformity*, 1902]. In this idea the concept of *weight of evidence* appears, which allows to consider that the *observed* frequencies are representative of the *actual* frequencies.

Quantitative induction, to Peirce, is by far the strongest way of inducing conclusions [27, 3/I: 183]. Firstly, its force comes from the fact that the procedure could be indefinitely extended, in a way that a objective probability relative to the occurrence of the pre-designated character can be defined in fact. Quantitative induction serves to measure probabilities in a precise way, and it *would* lead to a true answer:

Quantitative induction approximates gradually, though in an irregular manner to the experiential truth for the long run. The antecedent probable error of it at any stage is calculable as well as the probable error of that probable error. Besides that, the probable error can be calculated from the results, by a mixture of induction and theory. Any striking and important discrepancy between the antecedent and *a posteriori probable* errors may require investigation, since it suggests some error in the theoretical assumptions. But the fact which is here important is that Quantitative Induction always makes a gradual approach to the truth, though not a uniform approach. [23, 2.770, c.1905].

Quantitative induction, therefore, is a mathematical method used in determining the proportion of the distribution of the qualities between the members of a collection. The ascertainment of a statistical ration has this sole function: to distinguish the proportion of specific classes of events in relation to the whole of possible events.[8] If we think that induction will be indefinitely carried further on by a limitless community of inquirers, we have that with time given the method becomes gradually more reliable. Thus, quantitative induction is a safe method to prove the validity of hypothesis. Peirce, in a certain moment, says that for etymological reasons he prefers to call this kind of reasoning *adduction*, meaning the process of putting in discussion, the process of bringing to the center of the discussion the problematic cases and theories [27, 3/I: 190, letter to Kehler]. Remember that the idea of probable deduction means the deduction of a probability: through a deductive process, a certain probability is proved certain. In quantitative induction a similar process happens, with the difference that the new knowledge suggested is confirmed by abduction, just because *adduction* consists in taking the hypothesis further on, putting it forward, that is, it consists in advancing knowledge, by the application of the hypothesis to future cases:

The induction adds nothing. At the very most it corrects the value of a ration or slightly modifies a hypothesis in a way which had already been contemplated as possible.

Abduction, on the other hand, is merely preparatory. It is the first step of scientific reasoning, as induction is the concluding step. [26, 2: 752].

With this interpretation of the inductive method, Peirce can present an account of scientific inquiry whose chief merit lies in explaining how science advances combining moments of interruption with continuity. Quantitative induction is only one kind of induction, and induction proper is characteristic of only one stage of inquiry. From this point of view, the experimental test of hypotheses is a way of monitoring scientific procedure as a whole, so that the self-correctness of induction may be stretched out to the whole process. In other words, scientific method proceeds to use statistical instruments to ascertain a probabilistic truth, which is the only truth possible of being ascertained [32, chap. 1 *passim*], [8, p. 116].

The difference between abduction and induction is crucial. While the first has as its starting point the facts, and it seeks a theory to explain them, induction, on the contrary, starts from an explanatory hypothesis to seek the facts to support it:

Abduction makes its start from the facts, without, at the outset, having any particular theory in view, though it is motivated by the feeling that a theory is needed to explain the surprising facts. Induction makes its start from a hypothesis which seems to recommend itself, without at the outset having any particular facts in view, though it feels the need of facts to support the theory. Abduction seeks a theory. Induction seeks for facts. [26, 2: 752].

For this reason, the inductive and the abductive methods differ one from another being each one the inverse of the other, in an analogous way as the syllogistic forms of hypothetical and inductive reasonings were opposed in Peirce's

[8] As a matter of fact, the application of mathematical induction in the operation of distinguishing denumeral from abnumeral collections works as a criterion to distinguish classes of potency á 0, i.e., capable of being defined by the series of cardinal numbers, from classes of higher potency. Cf. [22] chap. 4, passim; [4], pp. 353–354.

early writings. Here, specifically, the logical forms of hypothesis and induction are not opposed, but inductive reasoning and the very process of imagining hypothesis: "In abduction the consideration of the facts suggests the hypothesis. In induction the study of the hypothesis suggests the experiments which bring to light the very facts to which the hypothesis had pointed." [*id.*].[9]

To Peirce, there is no strictly infallible knowledge, there is only a very high degree of probability that certain theories will continue to foresee the course of events. In effect, his account of the history of scientific thought is centered on the idea of probability, that is, in the idea that it is possible to ascertain a percentage of truth for real synthetical general propositions about the future, grounded upon the presently available evidence [27, 3/I: 139, *Probability,* c. 1903].

From this idea, it is possible to claim as a further consequence that no revolution in science happens purely and simply by itself, out of the blue, so to say. Instead of supposing facts are incomprehensible, it is needed to suppose that, notwithstanding they are striking, they reconcilable with what is known about other facts. In a certain moment, the quantity of strinking facts—*unforeseen* facts—will be so much that it will force a modification of the theory [26, 2: 751]. Even if enough evidences are not gotten, so as to demand a modification of the conceptual scheme, there is no possible return to the same point as before; either because the striking facts, if they happen again, will then not be as striking as at first anymore, and they will be in a certain way already incorporated to the theory, or because they have already led to the search for another similar facts, to gather enough evidence to abandon the theory, for instance. In fact, there are two ways of suggestion whereby abduction and induction make knowledge advance:

> The mode of suggestion by which, in abduction, the facts suggest the hypothesis is by *resemblance*,—the resemblance of the facts to the consequences of the hypothesis. The mode of suggestion by which in induction the hypothesis suggests the facts is by *contiguity*,—familiar knowledge that the conditions of the hypothesis can be realized in certain experimental ways. [26, 2: 752–753].

Therefore, to know is also to recognize. In every field of inquiry and investigation there is the tension between tradition and change, between invention and recognition. Anyway, the confrontation with experience is the motive for modifications, whether in the conceptual re-framing of theories already existent, to accommodate new facts, whether in the invention of new theories. However, the confrontation with experience does not say exactly *what* must be done, not even *how* it should be done. This is a decision we ourselves have to take—a pragmatic decision, which has to be based in the interactive context of the world surrounding us. From this standpoint, logic and ethics have a lot to learn from each other.

[9] For more on the heuristic role of abduction, see [14]. Ibri traces Peirce's abduction back to Kant's schematism of the categories in the Critic of the Pure Reason. It is impossible to develop this relation here, for it would require at least a whole new paper, one which we hope to be able to write in a further occasion.

4. Reasoning and Discovery

Now, it is very important to notice that Peirce's account of the way deduction, induction and abduction intermingle themselves as stages of the scientific method of discovery poses some very intriguing questions. By saying that even deduction can be a form of *probable* reasoning, for the reasoner may miss something, or some error may be hidden and the reasoner doesn't notice it, etc., Peirce makes room for questioning the very nature of logical necessity. In fact, we can say that, according to him, there is no problem at all with logical necessity in deduction; the reasoner has to correct his or her errors and if inquiry continues by correct methods, this correction will come sooner or later, maybe not for the present reasoner, but in the future. Well, that said, we need to say something about Peirce's conception of deduction: it is not that simple. For one thing is to proceed in a formal way without errors; a machine could be devised to do that. But another completely different thing is when we leave the realm of pure ideas and come to the real world of facts, and that's the more important point, since later in his life, as we said, Peirce came to conceive deduction, induction and abduction as phases os scientific inquiry. The issue of the relationship between the three forms of inference becomes then an issue of devising onto-logical formal models and of parametrically testing them. Peirce was not unaware of that. To this problem, he tried to provide an answer by distinguishing two kinds of deduction that deserve special attention: theorematic and corollarial deduction. This issue, in its turn, leads to several other very intriguing ones. We will not deal with the subject in detail, but will try to provide an overview of the matter in such a way as to point to future developments.

4.1. Theorematical and Corollarial Deduction

The problem is best understood if we consider the relation between mathematics and physics. Peirce says:

Pythagoras may be said to have originated the whole science of physics by observing a connection between the intervals of the tones of strings and the weights which stretched them. This probably belonged to the secret doctrine; for as it has come down to us, it is so totally wrong that the least experiment would show it. Yet without experiment the idea could not have arisen. Namely the statement made is that the ratios of weights 12:6, 12:8, 12:9 are, respectively, an octave, a fifth, and a fourth. Now the true ratios are precisely the square roots of these. Evidently, Pythagoras must have known the truth. It is a historical fact then that he was the father of physics. No small glory that.

[...]

Pythagoras thought that numbers were the substance of things. What he meant, I do not believe he knew or thought he knew. It was his highest *aperçu*. He felt he could not quite grasp it. [26, 1: 176–177].

Peirce sought to unite all these elements in a conception of mathematics as a science of discovery. What he himself thought of Pythagoras' thought maybe is the following:

A *state of things* is an abstract constituent part of reality, of such a nature that a proposition is needed to represent it. [. . .]

A *mathematical form* of a state of things is such a representation of that state of things, without definitely qualifying the subjects of the samenesses and diversities. It represents not necessarily all of these; but if it does represent all, it is the *complete* mathematical form. Every mathematical form of a state of things is the complete mathematical form of *some* state of things. [24, 2: 378, *The Basis of Pragmaticism in the Normative Sciences*, 1906].

From the quotation above we can grasp the aim of mathematical inquiry: to represent in a general and abstract manner all possible forms of states of things, no matter whether existent or not. This is what we called above the devising of onto-logical formal models and of parametrically testing them. According to such idea, we can quote the following Peircean definition of mathematics:

The first is mathematics, which does not undertake to ascertain any matter of fact whatever, but merely posits hypotheses, and traces out their consequences. It is observational, in so far as it makes constructions in the imagination according to abstract precepts, and then observes these imaginary objects, finding in them relations of parts not specified in the precept of construction. [23, 1.240, *A Detailed Classification of the Sciences*, 1902].[10]

In other words, the mathematician is not concerned with the *positive* truth of what in fact is, but only with his or her *hypothetical* truth, that is, with what could or could not be necessarily concluded from the imaginary hypotheses constructed. Mathematics, therefore, is the science that seeks to define pure possibilities. The mathematician first frames hypotheses, and next observes what necessarily it is possible to conclude as consequence from such constructions. After that, it is possible to generalize the conclusions obtained to every occasion possible of being described in the terms of the imagined hypotheses. Mathematical knowledge, thus, is purely hypothetical:

Modern science, even from the first, took away from demonstratively certain knowledge much of its luster; and mathematicians who alone produce such knowledge, now see clearly that such knowledge can only be knowledge of hypothetical states of things, or say of the implications of arbitrary hypotheses; and never can be *positive* science, that is, science of the real.[11]

Therefore, the imaginary constructions of mathematics can be applied to any situation of fact, any actual occasion, because they can be applied to *some* situation of fact.

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[10] See [19] p. 143. For the interpretation of Peirce's philosophy of mathematics, we basically follow [12] p. 192 ff.; [3]; and [21], specially chap. 12: "Pure Mathematics". For a detailed discussion of Peirce's philosophy of mathematics as well as his mathematics, the reader can consult [29] pp. 1–54, the introduction written by Ketner and Putnam; besides, see [22], chapters 3: "The Mathematics of Logic: formal aspects of the categories" and 4: "Infinity and Continuity"; cf. also [13, 15, 17, 20, 37]. For Peirce's classification of the sciences, the obligatory reference is [16]; but see also [31].

[11] This is taken from the manuscript numbered 283, p. 155, that has been partially published (untill page 151) in [24], vol. 2, and entitled "The Basis of Pragmaticism in the Normative Sciences", c. 1906.

Already in 1885, in *On the Algebra of Logic: A Contribution to the Philosophy of Notation*, Peirce recognized a difficulty in defining the scientific status of mathematics:

It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. [30, 5: 164].

Shortly after, Peirce tells the key to solve this dilemma, to wit, a correct understanding of the nature of deduction. We have seen that deduction is the only form of necessary reasoning; and also that, by changing the order of premisses and conclusions of this logical form, Peirce came to define the logical forms of inductive and hypothetical reasonings. In mathematics, he distinguishes two kinds of deduction, called theorematical and corollarial:

My first real discovery about mathematical procedure was that there are two kinds of necessary reasoning, which I call the Corollarial and the Theorematic, because the corollaries affixed to the propositions of Euclid are usually arguments of one kind, while the more important theorems are usually of the other. The peculiarity of theorematic reasoning is that it considers something not implied at all in the conceptions so far gained, which neither the definition of the object of research nor anything yet known about could of themselves suggest, although they give room for it. Euclid, for example, will add lines to his diagram which are not at all required or suggested by any previous proposition, and which the conclusion that he reaches by this means says nothing about. I show that no considerable advance can be made in thought of any kind without theorematic reasoning. When we come to consider the heuretic part of mathematical procedure, the question how such suggestions are obtained will be the central point of the discussion. [27, 4: 49].

Now, "reasoning essentially consists in the observation that where certain relations subsist certain others are found" [30, 5: 164]. What the distinction between the two forms of deductive reasoning shows is that mathematical reasoning is not only the observation of what is evident in a formal representation of a state of things, but it is also a constructive activity of such representations, by means of observing and modifying other representations. This is the heuristic part of mathematics, the one that makes us see something not implied in the premisses, clearly involving an abductive reasoning [7, p. 465 ff.]. We can take yet another passage, wherein the difference between the two kinds of deduction is expressed in other terms:

A Corollarial Deduction is one which represents the conditions of the conclusion in a diagram and finds from the observation of this diagram, as it is, the truth of the conclusion. A Theorematic Deduction is one which, having represented the conditions of the conclusion in a diagram, performs an ingenious experiment upon the diagram, and by the observation of the diagram, so modified, ascertains the truth of the conclusion. [24, 2: 298, *Nomenclature and Divisions of Triadic Relations*, 1903].

Mental experiments, according to the quotation above, are the same as observations of the diagram. In corollarial deduction, the procedure starts from the observation of a diagram such as it is, without any modification, to affirm the conclusion. The conclusion, therefore, is necessarily obtained only from what is expressed in the diagram, without any further adjunction to the conclusion. Theorematical deduction, in turn, modifies the diagram to discover new relations not evident in its initial form of presentation. Thus, it is the process whereby the truth of mathematical conclusions is ascertained "by performing a variety of experiments in our imagination." [27, 4: xiv, undated]. Now, if transformations are allowed in mathematical experiments besides the purely necessary extraction of conclusions, we have a very strong reason to think of deduction also as a creative and ampliative form of reasoning. In fact, this distinction between the theorematical and the corollarial makes all the more difficult for Peirce to maintain his early definitions in a rigid way. If in his youth he thought it could be possible a neat distinction between the three kinds of reasoning by way of defining their logical forms, it seems that later on he came to realize that the intermingling of the different forms of reasoning is much more interesting, from the heuristic point of view, than the precise definition of their logical boundaries. This does not mean, though, that Peirce became dismissive or blithe concerning definitions. A closer look into his way of defining mathematics, and of distinguishing it from logic, may show us how careful he was with his concepts and how acutely and profoundly he devised them.

4.2. The Difference Between Mathematics and Logic

Such definitions of mathematics in Peirce's works abound. One of them says the following:

The first [science of discovery] is mathematics, which does not undertake to ascertain any matter of fact whatever, but merely posits hypotheses, and traces out their consequences. It is observational, in so far as it makes constructions in the imagination according to abstract precepts, and then observes these imaginary objects, finding in them relations of parts not specified in the precept of construction. This is truly observation, yet certainly in a very peculiar sense; and no other kind of observation would at all answer the purpose of mathematics. [23, 1.240].

The quotation above shows very well the link between the theorematical and the corollarial processes in mathematical reasoning. Once more there appears the idea that the mathematician is not concerned with the positive truth of what actually is, but only with what could or could not be necessarily concluded, from the imaginary hypotheses framed. By showing the intertwinement of the moments of creation of formal models and of deduction of the conclusions necessarily implied in such models in the mathematical procedure, Peirce relates two distinct ways of defining mathematics:

[...] it is an error to make mathematics consist exclusively in the tracing out of necessary consequences. For the framing of the hypothesis of the two-way spread of imaginary quantity, and the hypothesis of Riemann surfaces were certainly mathematical achievements.

Mathematics is, therefore, the study of the substance of hypotheses, or mental creations, with a view to the drawing of necessary conclusions. [27, 4: 268, *On Quantity*, c. 1896].

In this way, Peirce follows the definition of mathematics given by his father Benjamin Peirce, according to which mathematics is the science *that draws* necessary conclusions, in contradistinction to logic, which is the science *of drawing* necessary conclusions:

The philosophical mathematician, Dr. Richard Dedekind, holds mathematics to be a branch of logic. This would not result from my father's definition, which runs, not that mathematics is the science *of drawing* necessary conclusions—which would be deductive logic – but that it is the science which *draws* necessary conclusions. [23, 4.239, *Minute Logic*, 1902].

Notwithstanding, if we focus on the definition of mathematics, we will be able to conclude that, in a certain sense, mathematics *is* logic, or at least, that logic is a constitutive part of the mathematical procedure. In fact, the most important difference between logic and mathematics is in the interest of each science. Take for instance the following passage:

For my part, I consider that the business of drawing demonstrative conclusions from assumed premisses, in cases so difficult as to call for the services of a specialist, is the sole business of the mathematician. Whether this makes mathematics a branch of logic, or whether it cuts off this business from logic, is a mere question of the classification of the sciences. I adopt the latter alternative, making the business of logic to be analysis and theory of reasoning, but not the practice of it. [23, 4.134, *The Logic of Quantity*, 1893].

On the one hand, the logician is not concerned with this or that special hypothesis, unless that in studying it, it brings him some new information on the nature of reasoning. On the other hand, the primordial interest of the mathematician is focused on the hypotheses taken individually, and in how it is possible to pass necessarily from the premisses to the conclusions in each case; the interest of the mathematician, therefore, is in the effectiveness of the methods of reasoning, for their capacity of being extended to other unknown instances; mathematics deals with the possible generalization of the hypotheses, rather than with sinuosities of reasoning. For instance, Peirce says:

The logician does not care particularly about this or that hypothesis or its consequences, except so far as these things may throw a light upon the nature of reasoning. The mathematician is intensely interested in efficient methods of reasoning, with a view to their possible extension to new problems; but he does not, *qua* mathematician, trouble himself minutely to dissect those parts of this method whose correctness is a matter of course. [23, 4.239].

This point is clarified if we look up at how each scientist considers logical algebra:

The mathematician asks what value this algebra has as a calculus. Can it be applied to unraveling a complicated question? Will it, at one stroke, produce a remote consequence? The logician does not wish the algebra to have that character. On the contrary, the greater number of distinct logical steps, into which the algebra breaks up an inference, will for him constitute a superiority of it over another which moves more swiftly to its conclusions. He demands that the algebra shall analyze a reasoning into its last elementary steps. Thus, that which is a merit in a logical algebra for one of these students is a demerit in the eyes of the other. The one studies the science of drawing conclusions, the other the science which draws necessary conclusions. [*Id.*].

So, logic seems to be more interested in the "rhetorical" character of reasoning, aiming at explicating *every* step—and not only the necessary ones—of reasoning from premisses to conclusions. Indeed, rhetoric is an essential part of the Peircean conception of logic. Since the method of science proceeds pragmatically and experientially, scientific activity necessarily involves the adoption of methods of inquiry that are public and dialogical. And this, of course, leads to an amplification of the domain of rhetoric in the context of inquiry. Thus, logic is widely conceived by Peirce as semiotics, "the quasi-necessary, or formal, doctrine of signs" [23, 2.227], see also [24, 2: 327]. This is a point we cannot develop here, for it alone would require much more than just one article; we will return to the difference between mathematics and logic in the conclusion. For now, let us quote just one passage where this idea of logic is made clear, while at the same time acknowledging this insufficiency:

All thought being performed by means of signs, logic may be regarded as the science of the general laws of signs. It has three branches: (1) *Speculative Grammar*, or the general theory of the nature and meanings of signs, whether they be icons, indices, or symbols; (2), *Critic*, which classifies arguments and determines the validity and degree of force of each kind; (3), *Methodeutic*, which studies the methods that ought to be pursued in the investigation, in the exposition, and in the application of truth.[12] [24, 2: 260, *An Outline classification of the Sciences*, 1903].

These three divisions of logic constitute what Peirce calls the philosophical *trivium*,wherein the rhetoric purport of each of them is manifest, since the scope of logic is broadened so to include the study of the *modes of signifying*, in general [24, 2: 19, *Of Reasoning in General*, 1895]. Now, according to the quotation above, the study of the forms of reasoning that are also characteristic of the different phases of scientific inquiry—deduction, induction, abduction or retroduction, as the quotations we used say—falls within the scope of Methodeutic, which is different from mere methodology, since it is not composed of a set of norms to be applied to every investigation whatsoever. Peirce does not want to provide a universal method that would solve all the puzzles; rather, he wants to differentiate with conceptual rigor and logical clarity the different roles each logical procedure play in scientific inquiry. So, we can say that the role of investigating and searching is properly played by abductive or retroductive reasoning; the role of exposing what is thus found is properly played by inductive practices; and the correct application of the general results achieved is provisionally ascertained by deductive reasoning. As we saw, deductions can

[12] For a good succinct introduction to the theme of rhetoric in Peirce's thought, see [18].

strictly follow what is affirmed in the premisses or can modify the premisses, in case of which we can see different possible relations. And then the whole process begins again, and there is no rigid temporal separation between the three stages: they are all concomitant.

In contrast to logic, mathematics rests upon the principle of parsimony in its procedures, what is just the application of Ockham's razor to reasoning. It is not the business of mathematicians to seek to evaluate or to classify reasonings, developing all the steps, saying which is the beautiful reasoning, or the most effective; mathematics, as an experimental science over diagrams, seeks only to study the possible hypothetical consequences, in a tied relation with what the pragmatic maxim declares:

Consider what effects, which might conceivably have practical bearings, we conceive the object of your conception to have. Then, our conception of these effects is the whole of our conception of the object. [30, 30, 3: 266, *How to Make Our Ideas Clear*, 1878].

The definition of mathematics as the study of what is true of hypothetical state of things is more frequent in Peirce's writings. Such definition justifies why the truth values of mathematical sentences do not matter so much: the mathematician admits as object of study any hypothesis, without being interested in knowing whether they are true or not. Very often the mathematician constructs a mathematical form following the indications given to him or her by other scientists, who find themselves in an aporetical situation without understanding the relations the objects entertain in a certain state of things:

Now a mathematician is a man whose services are called in when the physicist, or the engineer, or the underwriter, etc., finds himself confronted with an unusually complicated state of relations between facts and is in doubt whether or not this state of things necessarily involves a certain other relation between facts, or wishes to know what relation of a given kind is involved. He states the case to the mathematician. The latter is not at all responsible for the truth of those premisses: that he is to accept. The first task before him is to substitute for the intricate, and often confused, mass of facts set before him, an imaginary state of things involving a comparatively orderly system of relations, which, while adhering as closely as possible or desirable to the given premisses, shall be within his powers as a mathematician to deal with. This he terms his *hypothesis*. That work done, he proceeds to show that the relations explicitly affirmed in the hypothesis involve, as a part of any imaginary state of things in which they are embodied, certain other relations not explicitly stated. [27, 4: 267].

Hence, it makes no sense to distinguish sharply between pure and applied mathematics [37, pp. 41–45]. In fact, there is no point in distinguishing between pure and applied *knowledge* at all. Once the mathematician constructs hypotheses grounded upon a suggestion from experience, he or she has in sight some application of the models devised—we can generalize this for all knowledge, for all thought, for the intellectual power of imagining abstract formal schemes is one and the same as the intellectual power of conceiving their consequences and of testing them against experience. In sum, there is no separation between distinct uses of reason, the practical and the speculative, as there was for Kant, for instance. Again we find, in this point, Peirce's pragmatism as a scientific method, in fact, as the true logic of abduction [24, 2: 224 and 234–235, 1903]. Let us consider the following passage:

It is now generally admitted, and it is the result of my own logical analysis, that the true maxim of abduction is that which Auguste Comte endeavored to formulate when he said that any hypothesis might be admissible if and only if it was verifiable. Whatever Comte himself meant by verifiable, which is not very clear, it certainly ought not to be understood to mean verifiable by direct observation, since that would cut off all history as an inadmissible hypothesis. But what must and should be meant is that the hypothesis must be capable of verification by induction. Now induction, or experimental inquiry, consists in comparing perceptual predictions deduced from a theory with the facts of perception predicted, and in taking the measure of agreement observed as the provisional and approximative, or probametric, measure of the general agreement of the theory with fact.

It thus appears that a conception can only be admitted into a hypothesis in so far as its possible consequences would be of a perceptual nature; which agrees with my original maxim of pragmatism as far as it goes. [24, 2: 225, *The Nature of Meaning*, 1903].

All thought, always projected to the future, leads to the formation of convictions and habits, which are settled within a horizon of experience and expectation. In effect, we also saw that to each kind of inference there corresponds a logical modality: the business of deduction is to establish necessary reasonings, that of induction to establish probable reasonings, that of abduction is to delimitate an expectation. All three are put to work together when we try to discover something:

What, then, is the end of an explanatory hypothesis? Its end is, through subjection to the test of experiment, to lead to the avoidance of all surprise and to the establishment of a habit of positive expectation that shall not be disappointed. [25, p. 250].

Furthermore, there is a demand that the mathematician be capable to imagine models simple enough to work with them. Then, it is his or her task to simplify the relations to the most to try to find those that are the most elementary. There is no obtrusive metaphysical reason behind this; as the mathematical procedure is marked by the parsimony of reasoning, the safest way of avoiding mistakes is also the easiest way to discover necessary relations, to wit, to simplify and to reduce the relations to the absolutely essential. Thus, what the mathematician does is to imagine a highly abstract model of simplified relations, yet still capable of expressing the relations of the facts. This high degree of abstraction allows for the generalization:

All features that have no bearing upon the relations of the premisses to the conclusion are effaced and obliterated. The skeletonization or diagrammatization of the problem serves more purposes than one; but its principal purpose is to strip the significant relations of all disguise. [23, 3.559, *The Logic of Mathematics in Relation to Education*, 1898].

The mathematician, then, makes two different things. First, he or she imagines a hypothesis, represented in the form of a highly abstract diagram of the state of things, representative only of its most essential relations, without caring about whether the representation will or will not correspond to the actual reality. Second, the mathematician begins to draw necessary conclusions from such relations, conclusions not explicit in the diagram. Mathematical necessity, therefore, comes from the logical connection settled between premisses and conclusions. The mathematician adopts hypotheses, conclusions, rules, and goes on to verify which state of things necessarily follows from another. Mathematics thus defined is purely formal, and concerns only the possibility of application of its models to the actual reality:

> Now the feature of mathematics which separates it widely both from Philosophy and from every special science is that the mathematician never undertakes (*qua* mathematician) to make a categorical assertion from the beginning of his scientific life to the end. He simply says what *would be* the case under hypothetical circumstances. [27, 4: 208, *Reason's Conscience*, 1904].

The hypothetical character of its assertions, besides distinguishing it from the positive sciences that state factual truth, warrants the very necessity of the conclusions: the interest of the mathematician is uniquely for the form of relations. Mathematics, consequently, opens a vast field of structural possibilities [13, p. 4], [17, p. 79]. The special circumstance embodying what was mathematically ascertained is merely contingent. To know if a given form can *de facto* be applied to an actually existent state of things is a scientific question that each scientist must resolve according to his or her needs. The mathematician only defines the *de jure* question, that is, to the mathematician is due solely the work with structures likely to be applied, which result in certain necessary conclusions [17, p. 79].

It is important to notice the possibility of deducing necessary conclusions from mathematical propositions; this is the very nature of mathematical inquiry:

> The first of these [three divisions of heuretic science] comprises the business of finding out what might and more especially what could not, be true under described circumstances, without asking whether or not such circumstances ever really occur. To my apprehension, this precisely defines *mathematics* [...].[13]

The meaning of mathematical terms and propositions, in this way, is confined to their form of expression in mathematical signs: "The meaning of a mathematical term or sign is its expression in the kind of signs in the imaginary or other manifestations of which the mathematical reasoning consists. For geometry, this [expression] is [in] a geometrical diagram." [27, 2.251]. However, this does not mean that mathematical truths are defined by their use in determined contexts, or that they are determined by some linguistic convention. It is the importance of the *modus operandi*, that is, the way how

[13] This is taken from the unpublished manuscript 1338, p. 6, ca. 1906.

demonstrations are made and the application of the very mathematical demonstrative procedures to the hypothetical diagrams that gives mathematics its sureness:

I certainly think that the certainty of pure mathematics and of all necessary reasoning is due to the circumstance that it relates to objects which are the creations of our own minds, and that mathematical knowledge is to be classed along with knowledge of our own purposes. [25, p. 227].

The meaning of mathematical constructions is not given *ab ovo,* but it is defined by demonstration: to reason is not only to use meanings, it is not merely to operate signs, but it is also to construct them, to manipulate signs in a certain way to determine them and to suggest certain interpretations.[14]

The Peircean definition of mathematics, in short, has two main characters; to wit:

1st, mathematics does not concern a special range of entities, as every other science depending upon it in the classification of the sciences. In other words, it is not defined neither by means of the specificity of its objects, nor by the nature of its propositions, nor even by the kinds of truths it may exhibit; mathematics has nothing to say about the truth of fact because it is a science dealing with hypothesis and abstractions; in a more traditional language one could say that mathematics is a science, the objects of which are *entia rationis* [24, 2: 352], see [37, pp. 30–31].

2nd, according to his father Benjamin Peirce, mathematics is the science that draws necessary conclusions. Indeed, every necessary reasoning is mathematical reasoning [27, 4.47]. This second characteristic raises the problem of the definitions of a mathematical ontology.[15] What would be the nature of such *entia rationis*? Would they be purely arbitrary conventions, since they do not refer to actual reality? Or would they be purely analytical propositional systems, or even tautological systems? If it be so, why then to insist in the practical side of mathematics, that is, over the possibility of application of mathematics to problems of the positive sciences?[16] In short, this is a question of how to connect Peirce's indeterminist realism with his conception of mathematics. As a matter of fact, one can ask whether fallibilism in mathematics is really problematic for Peirce, who several times asserts that error in mathematics is due only to a blunder in reasoning [23, 1.149, *c.* 1897; 7.108, 1892; 1.248, 1902], and [27, 4: 210, 1904]. Therefore, it is not difficult for him to think of the necessity of mathematics as perfectly compatible with an ideal system, wherein one can reason about possible cases (and, therefore, undetermined instances), and also about actual cases.

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[14] As a matter of fact, this is linked to the theme of self-controlled thinking, which is an essential subject for the definition of logic as normative science. See [37] p. 34.

[15] For the supposed Platonism of such conception, a subject we are not able to deep here, see [37] and [17].

[16] See how Tiercelin [37, p. 31] presents the problem: "If one accepts the notion of applied mathematics as something which is needed by all sciences, what is to warrant that such idealizations have the objective validity which justifies their being used by these other sciences?"

5. Outlines for a Comparison Between Peirce and Whitehead on the Nature of Mathematics

Finally, it is time now to justify the epigraph. Let us first say that is not the main purpose of this article to develop the relation between Peirce and Whitehead. This would require far more than just one paper. The purpose of the epigraph and of this conclusion is to suggest a way for further developments in research, for we believe that the general spirit of Whitehead's claim that science makes an aeroplane flight in thin air of imagination just to land again in the rough ground of particular observation is shared by Peirce in his theory of inquiry. All this can be clarified if we attend to Peirce's and Whitehead's philosophies of mathematics in their similar aspects and differences as well. So, all that is put forward here is intended as a first approximation and a beginning, not more than that. We hope this sketch of a comparison between the two thinkers may add to what has been hitherto discussed and point to further interesting developments.

Whitehead's quote at the beginning of this paper seems to represent very well the general idea behind Peirce's theory of inquiry. But not only: when we come to a closer examination of specific points, there are similarities as well.

The initial point should then is a very basic one that recovers the general idea. For Peirce, a very basic thing to understand is what reasoning consists in: "reasoning consists in the observation that where certain relations subsist certain others are found" [24, 1: 227, *On the Algebra of Logic*, 1885]. This is the very nature of inference: from the verification that there are certain subsisting relations, we go on to assert that some others also subsist, linking the latter with the former with more or less necessity in our reasoning (that is, deductively, inductively, abductively). For Peirce, this passage is made easier when we contemplate a diagram of some sort that reveals to us the simultaneity of the relations; when we connect them in reasoning, we pass to the linearity of the relations, thinking in terms of what relations come before or after which others. A full development of the issue of the temporality of the process would lead us far away from the limits of this paper. The point we want to stress is the contemplation of diagrams, which are for Peirce instances of icons: "I call a sign which stands for something merely because it resembles it, an *icon*. Icons are so completely substituted for their objects as hardly to be distinguished from them. Such are the diagrams of geometry" [24, 1: 226]. The point with diagrams is that they are completely abstract, resembling objects only through their form. In other words, there is an isomorphic homology between the diagram and the object, so that various other objects of the same form can be resembled by the same diagram or icon. This is an essential feature of the abstractness of mathematics and understanding it makes it possible to answer why mathematics is simultaneously deductive and makes us discover something new, for

> . . . all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of

the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. For instance, take the syllogistic formula,

All M is$P$
S is M
$\therefore S$ is P.

This is really a diagram of the relations of S, M, and P. The fact that the middle term occurs in the two premisses is actually exhibited, and this must be done or the notation will be of no value. [24, 1: 227–228].

Let us briefly return to the discussion about the difference between mathematics and logic to notice that Peirce says *all deductive reasoning* involves observation; so, logic and mathematics both are observational sciences and this fact does not contradict their apodictic natures. This emphasis on observation is necessary to explain why they afford for new discoveries; the difference is that the logician's business is to study necessary reasoning, and the mathematician's is its practice [21, p. 230-231]. That is why Peirce equates mathematical reasoning with necessary or deductive reasoning *par excellence*, for that is, as aforementioned, what a mathematician does: mathematics is the science that draws necessary conclusions.

Now, it seems clear to us that Whitehead adopts a conception of mathematics that is substantially almost the same as Peirce's in some respects. Take, for instance, the following statement:

The point with mathematics is that in it we have always got rid of the particular instance, and even of any particular sorts of entities. So that, for instance, no mathematical truths apply merely to fish, or merely to stones, or merely to colours. So long as you are dealing with pure mathematics, you are in the realm of complete and absolute abstraction. All you assert is, that reason insists on the admission that, if any entities whatever have any relations which satisfy such-and-such purely abstract conditions, they must have other relations which satisfy other purely abstract conditions. [39, p. 31].

The first respect of proximity is the highest abstraction possible attained in mathematics. For Whitehead, "Mathematics is thought moving in the sphere of complete abstraction from any particular instance of what it is talking about." [39, p. 32]. There is perfect harmony between Whitehead and Peirce, when the latter claims: "[Abstraction] may be called the principal engine of mathematical thought." [23, 2.364, *Quantity*, 1902]. Mathematical notions, such as collections and numbers, are the outcomes of abstraction, which cannot be confounded with generalization:

[Abstraction] consists of seizing upon something which has been conceived as a *ἔπος πτερόεν* [winged word], a meaning not dwelt upon but through which something else is discerned, and converting it into an *ἔπος απτερόεν* [non-winged word], a meaning upon which we rest as the principal subject of discourse. Thus, the mathematician conceives an operation as something itself to be operated upon.[17] [23, 1: 83].

When operations are submitted to other operations in mathematics, what is done is an abstraction. The abstractive operation renders possible to take an object as the subject upon which experiments are made, and from it to infer conclusions about other objects [36, p. 296]. For instance: "A particle is somewhere quite definitely. It is by abstraction that the mathematician conceives the particle as occupying a *point*." [27, 4: 11]. Abstraction, then, defines a fixed signification—a *word without wings* – to serve as bedrock for the understanding of other objects. It is, in fact, an operation for isolating general relations, from which we can draw conclusions, and this operation can be made in at least two ways.

According to Peirce, there are two elementary kinds of abstraction, precisive and hypostatical:[18]

With this preface, we may proceed to consider *hypostatic abstraction*; that is, abstraction in the sense in which we speak of abstract nouns, as contradistinguished from *precisive* abstraction, which consists in concentrating attention upon a particular feature of a supposed state of things. [26, 2: 739, *On the Logic of Drawing History*].

Precisive abstraction, as it is clear, is only an act of attention, in which certain aspects are noticed, others neglected. In hypostatical abstraction, an individual object is taken as an *ens rationis*, that is, an entity whose being consists in some other fact; its logical peculiarity is in that the subject of the conclusion is not expressed in the premisses, and yet the conclusion remains necessary [23, 4.463, *On Existential Graphs, Euler's Diagrams, and Logical Algebra*, 1903]. Peirce's favourite illustration for hypostatical abstraction is taken from the third *intermezzo* of Molire's *Le Malade Imaginaire*. Molire describes an oral examination, wherein a doctor in medicine asks a graduate student which are "the cause and the reason" for opium putting people to sleep. Confident and full of certainty, the bachelor answers back in his best Latin: "Quia est in eo virtus dormitiva", that is, "Because there is in it a force that makes people sleepy." He is then applauded by the choir and accepted in the body of doctors. Molire was satirizing, criticizing the pretension to explain everything with beautiful though empty words, what in truth one does not

[17] The image of the winged words Peirce might have borrowed from Homer, who in the Iliad, book I, lines 197–204, describes the encounter of Achilles and Pallas Athena with these words: "Rearing behind him Pallas seized his fiery hair - / only Achilles saw her, none of the other fighters - / struck with wonder he spun around, he knew her at once, / Pallas Athena! The terrible blazing of those eyes, / and his winged words went flying: "Why, why now?/ Child of Zeus with the shield of thunder, why come now?/ To witness the outrage Agamemnon just committed? / I tell you this, and so help me it's the truth - / he'll soon pay for his arrogance with his life!".

[18] Do not confound hypostatical abstraction with the operation of prescision. See [30] 2: 50–51: On a New List of Categories, 1867. We will not explain the latter here. See [32], pp. 85 ff.

know how to explain. For Peirce, even such a declaration can provide some knowledge, since it asserts that there is an explanation for the fact, besides the very fact: "For it does say that there is *some* peculiarity in the opium to which sleep must be due; and this is not suggested in merely saying that opium puts people to sleep." [23, 5.534, *Pragmatism*; see 27, 4: 11]. In other words, hypostatical abstraction allows for the formulation of a general conception of a reality that even though it is manifested in the individual phenomena, it is not neither exhausted in it nor explicit:

> But *hypostatic* abstraction, the abstraction which transforms 'it is light' into 'there is light here,' which is the sense which I shall commonly attach to the word abstraction (since *prescission* will do for precisive abstraction) is a very special mode of thought. It consists in taking a feature of a percept or percepts (after it has already been prescinded from the other elements of the percept), so as to take propositional form in a judgment (indeed, it may operate upon any judgment whatsoever), and in conceiving this fact to consist in the relation between the subject of that judgment and another subject, which has a mode of being that merely consists in the truth of propositions of which the corresponding concrete term is the predicate. Thus, we transform the proposition, 'honey is sweet,' into 'honey possesses sweetness.' 'Sweetness' might be called a fictitious thing, in one sense. But since the mode of being attributed to it *consists* in no more than the fact that some things are sweet, and it is not pretended, or imagined, that it has any other mode of being, there is, after all, no fiction. The only profession made is that we consider the fact of honey being sweet under the form of a relation; and so we really can. [23, 4.235, *The Simplest* Mathematics, 1902].

Now, it is not merely to suppose *entia rationis*; hypostatical abstraction leads us to see non-evident relations, leads us to discover that the *virtus dormitiva* of opium must be actually real, since opium puts us to sleep. The *virtus dormitiva*, therefore, taken separately from the fact that opium makes people sleepy, is put as an entity—that is why such abstraction is called *hypostatical* [36, p. 296]. If this entity is real or not, this is a matter for scientific inquiry to discover, and then we have the whole process described above in the previous sections of this paper.

Let us notice that Whitehead highlights relations and abstraction more or less in the same way to define *pure* mathematics:

> For example, take the question of number. We think of the number 'five' as applying to appropriate groups of any entities whatsoever—to five fishes, five children, five apples, five days. Thus, in considering the *relations* of the number 'five' with the number 'three', we are thinking of two *groups of things*, one with five members and the other with three members. But we are entirely abstracting from any consideration of any particular *entities*, or even of any particular sorts of entities, which go to make up the membership of either of the two groups. We are merely thinking of those *relationships* between those two groups which are entirely independent of the individual essences of any of the members of either group. This is a very remarkable feat of abstraction; and it must have taken ages for the human race to rise to it. During a long

period, groups of fishes will have been compared to each other in respect to their multiplicity, and groups of days to each other. But the first man who noticed the analogy between a group of seven fishes and a group of seven days made a notable advance in the history of thought. He was the first man who entertained a concept belonging to the science of pure mathematics. [39, p. 30, our italics].

Now, for Whitehead 'abstract' means the following: "To be abstract is to transcend particular concrete occasions off actual happening. But to transcend an actual occasion does not mean being disconnected from it." [39, p. 228–229]. From the quotation above, we see that the connection lies in the fact that abstraction is an operation that has its origins in the common human experience of noticing analogies between different groups of actual things. To put it roughly, abstraction makes us think of the possible relations between other groups of things of the same form. And to understand this connection between the general and the particular is the chief aim of *any* science: "To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought." [38, p. 11]. Mathematics performs just that when, examining the events, we disregard all specificities of the particular things and keep only with "the positions and motions of the molecules, a description which ignores me and you and him, and also ignores sight and touch and taste and smell." [38, p. 13]. That is why, for Whitehead, all "explanations of the order of events necessarily tend to become mathematical" [38, p. 11], for mathematical ideas supply just this kind of abstraction required by science. In fact, the most general conceptions thus acquired extrapolate limits between science and philosophical speculation, for "metaphysical categories" themselves "are not dogmatic statements of the obvious; they are tentative formulations of the ultimate generalities." [40, p. 8]. We could say that, according to this rationale, mathematical conceptions—varied as they are: quantity, structure, infinity, simultaneity, continuity, linearity, multitude, zero, denumeral, abnumeral, and so forth—express in the most general way the relations observed between real entities. This is what defines pure mathematics for Whitehead, making it possible for him to state: "Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings." [39, p. 31]. And here we find a difference between the two thinkers: we have already seen that distinction between pure and applied mathematics in Peirce's thought is not an essential difference. Nonetheless, Peirce makes it very clear that mathematics is not defined as a science by some actual employment, a point which is very important for understanding the place he reserves for fallibilism in mathematics.

The second similarity we want to notice is that Whitehead's remark presents Peirce's very essence of reasoning: where certain relations occur, some others also do. And we get to understand this in the utmost abstraction, regardless of "any particular entities, or even of any particular sorts of entities". This leads us to the issue of necessity: in mathematics, as we are far away from particularities of the actual world, we deal only with "complete abstract generality", so that necessity is an essential characteristic mathematical reasoning. For instance:

In the pure mathematics of geometrical relationships, we say that, if *any* group entities enjoy *any* relationships among its members satisfying *this* set of abstract geometrical conditions, then such-and-such additional abstract conditions must also hold for such relationships. [39, p. 33].

That the necessary deductive nature of reasoning is operative in mathematics seems clear from the quotation above. But that is not all; there is space for uncertainty in the applications of mathematics, just as for Peirce:

The certainty of mathematics depends upon its complete abstract generality. But we can have no *a priori* certainty that we are right in believing that the observed entities in the concrete universe form a particular instance of what falls under our general reasoning. To take another example from arithmetic. It is a general abstract truth of pure mathematics that any group of forty entities can be subdivided into two groups of twenty entities. We are therefore justified in concluding that a particular group of apples which we believe to contain forty members can be subdivided into two groups of apples of which each contains twenty members. But there always remains the possibility that we have miscounted the big group; so that, when we come in practice to subdivide it, we shall find that one of the two heaps has an apple too few or an apple too many. [39, p. 33].

This also opens room for a sort of fallibilism in mathematics, and, just as for Peirce, this fallibilism has nothing to do with the logical necessity or nature of mathematical reasoning; error can emerge either from a mistaken assumption, or else at the moment of employment or application of the conclusions. For Whitehead, there are three steps we need to follow to see if there is some mistake in our reasoning: first, to review the whole reasoning, "to make sure that there are no mere slips in it"; second, to determine whether the premisses are exactly defined or if there is some hidden condition we introduced that does not hold; and third, whether "our abstract postulates hold for the particular case in question" or not. This where he states that the more complex and varied are the conditions of application of mathematics to matters of fact, the more difficult is the attainment of certainty: "in some simple instances, such as the counting of forty apples, we can with little care arrive at practical certainty. But in general, with more complex instances, complete certainty is unattainable." [39, p. 35]. So, he can conclude: "The only logical conclusion to be drawn, when a contradiction issues from a train of reasoning, it is that at least one of the premises involved in the inference is false." [40, p. 8].

Now, this point is also advocated by Peirce, stated in other words: there is nothing wrong with logical necessity in mathematical reasoning, but one may blunder. And when applying the conclusions mathematically reached, this blunder is likely to happen, so that we have to review the process and find where we did the mistake. In that way, we cannot *a priori* believe in the correctness of our calculus. This is indeed a very Peircean claim, for it leads one to think that there are rules of transformation which limit the operations and the scope of the reasoning, regardless of any observation or empirical employment of it [21, p. 230]. And, then, the problem can only be with the *application to* matters of fact. So, we find indeed an argument for epistemological fallibilism, for it is based on our lack of knowledge or on some inattention of our part. It is not an argument for any ontological fallibilism, for it does not claim to be any problem with the world or with reasoning itself.[19]

This leads to another point. Peirce also insists upon the unavoidable and abstract character of mathematical conclusions:

The mathematician does not "rely" upon anything. He simply states what is *evident*, and notes the circumstances which make it evident. When a fact is evident, and nobody does or can doubt it, what could "reliance" upon anything effect? [27, 4: 209, *Reason's Conscience*].

In this way, the evidence of the mathematical diagrams forcefully obliges the mathematician to recognize the necessity of the consequences, or, to use Whitehead's words, "reason imposes on the admission" of what is evident. And this admission is made in complete abstract terms, as we have seen.

Let us also notice Whitehead's emphasis on entities and particular things and sketch some very general remarks on mathematical ontology. Let us begin with a passage from Whitehead to remember the abstract nature of mathematical objects:

The first acquaintance which most people have with mathematics is through arithmetic. That two and two make four is usually taken as the type of a simple mathematical proposition which everyone will have heard of. Arithmetic, therefore, will be a good subject to consider in order to discover, if possible, the most obvious characteristic of the science. Now, the first noticeable fact about arithmetic is that it applies to everything, to tastes and to sounds, to apples and to angels, to the ideas of the mind and to the bones of the body. The nature of the things is perfectly indifferent, of all things it is true that two and two make four. Thus we write down as the leading characteristic of mathematics that it deals with properties and ideas which are applicable to things just because they are things, and apart from any particular feelings, or emotions, or sensations, in any way connected with them. This is what is meant by calling mathematics an abstract science. [38, p. 9].

Now, the possibility of application to whatever things and objects there are in the actual world is a result of the abstractness of mathematical ideas. This is not to say that mathematics is defined by this possibility; quite the contrary, its abstractness is what determines a wide range of applicative possibility. And this is a good explanation why, all aside from fact that there is no essential difference between pure and applied mathematics, as a science, mathematics is not defined by actual employments. Even if mathematical objects are devised by abstraction from *some* actual circumstance, its abstract nature

[19] To develop this point, more than another full article would be necessary. See note 1.

allows for its application to *any* circumstance of the same form.[20] This is a basic algebraic idea, according to Whitehead:

The ideas of *any* and of *some* are introduced into algebra by the use of letters, instead of the definite numbers of arithmetic. Thus, instead of saying that 2+3=3+2, in algebra we generalize and say that, if x and y stand for *any* two numbers, then $x+y = y+x$. Again, in the place of saying that $3 > 2$, we generalize and say that if x be *any* number there exists *some* number (or numbers) y such that $y > x$. [38, p. 15].

Peirce strongly emphasizes the necessity of mathematical conclusions as connected to its abstractness; but the only thing one may affirm concerning their applicability is that they *might* or *might not* be actualized. Taking algebra instead of arithmetic as an instance, Peirce states that algebra indeed does not want to say anything besides its own forms:

The algebraic system of symbols is a *calculus*; that is to say, it is a language to *reason in*. Consequently, while it is perfectly proper to define a *debt* as negative property, to explain what a negative quantity is, by saying that it is what debt is to property, is to put the cart before the horse and to explain the more intelligible by the less intelligible. To say that algebra means anything else than just its own forms is to mistake an *application* of algebra for the *meaning* of it. [23, 4.133].

In fact, the true mathematical objects are the very forms of relations, or, as N. Houser says, the "relational structures".[21] Generalizing the various relations we find in the actual world, giving them a substantive form, the relations abstracted from all accidentality become the objects of mathematical inquiry. Mathematics, then, acquires the status of a formal logic of relations. Maybe the most meaningful discovery of mathematics, from this point of view, be that there are three fundamental forms of relation: monadic (1), dyadic (2), and triadic (3). Here we have the famous Peircean thesis of the essential irreducibility of the triad, and the reducibility of all other higher relations ((4), (5), (6), etc.) to the triad. Triads combine both dyads and monads, and dyads combine monads, that is, (2) and (1) are present in (3), and (1) is present in (2). In the same way that a triad, however, cannot be reduced to a dyad, that is, a dyad cannot represent the same relations that a triad can; and every other relations of higher order than three can be reduced to the triad. Thus, in (4) there is (3) and (1), and in (5) there is (3) and (2), and in (9) there is ((3) x (3)], and so forth. [24, 364, *The Basis of Pragmaticism in Phaneroscopy*, 1906].

Monads, dyads, and triads make our set of fundamental categories of relations. The relation of one category containing another that is exhibited is better understood as the presence of the inferior relation in the superior one, in

[20] This is an important point that lies at the root of the distinction between actual and potential entities, which is to be developed in Process and Reality. But we cannot but point to this development here. For a short introduction to Process and Reality, see [9]; for a detailed account of this specific subject, see [35] especially chap. 1: "The actual entity".

[21] In what follows, we adopt N. Houser's interpretation of Peirce's method of derivation of the categories; see [13].

a structural manner, as if a part-to-whole relation. In examining a diagram, the mathematician *sees* that monads are structurally elementary, i.e., that they are *firsts*. Dyads, consequently, depend on monads, i.e., they necessarily contain them; and the same is true for triads as to dyads. The three essential forms of relation therefore correspond to being structurally first, second or third in mathematics. Through hypostatical abstraction, i.e., the abstraction that permits the passage from an individual to *entis rationis* [23, 4.235, *The Simplest Mathematics*, 1902; 4.549, *Prolegomena to an Apology of Pragmaticism*, 1906], the mathematician comes to the categories of firstness, secondness, and thirdness. This group of categories, because they are extremely formal, is applicable to any triads, whether possible or actual. Thus, we can know a priori which will be the form of experience, once it is possible to know what will always be the relations of dependence to be found in experience.

We can see another general convergence between the two thinkers on this point. In fact, Whitehead strongly claims for the abstract nature of mathematics in terms of *objectification*, a term which can be understood in the following way: "The term 'objectification' refers to the particular mode in which the potentiality of one actual entity is realized in another actual entity" [40, p. 23].

We can understand objectification in terms of the possibility to apply mathematical ideas to actual objects: fishes, apples, and so forth. Second, we can understand objectification in terms of construing ideal abstract objects, such as geometrical figures, for instance [38, p. 5]. For Whitehead, our knowledge of the facts of the world is gained through our sensations: we see, hear, taste, smell etc., ascribing the origin of these sensations to the "relations between the things that form the external world". This leads to a conception of the world as a whole wherein we exist, and not only of the world as a source of individual sensations, but one world where we exist together: "the world as one connected set of things which underlies all the perceptions of all people". The abstract ideal mathematical objects are the result of our attempt at describing the "relations between the things which form the external world" regardless of any dependence on any particularity of sensation. [38, p. 12]. But it is a completely another question whether these abstract ideas are mere *generalizations* of facts, for "the ideas, now in the minds of contemporary mathematicians, lie very remote from any notions which can be immediately derived by perception through the senses" [39, p. 29]. So, Whitehead concludes:

But when we have put aside our immediate sensations, the most serviceable part—from its clearness, definiteness, and universality—of what is left is composed of our general ideas of the abstract formal properties of things; in fact, the abstract mathematical ideas mentioned above. Thus it comes about that, step by step, and not realizing the full meaning of the process, mankind has been led to search for a mathematical description of the properties of the universe, because in this way only can a general idea of the course of events be formed, freed from reference to particular persons or to particular types of sensation. [38, p. 12].

In contrast to Whitehead, who puts a lot of stress in the relationship between the abstract mathematical "ideas" and "particular entities" and "groups" of ideas and groups of entities, Peirce does not understand mathematical relations as abstract relations of *sets of things*, whatever might be [39, p. 30, our italics].

Whitehead speaks in terms of groups of particular things; in fact, he even speaks of the world as we imagine it as "one connected set of things", as we have just seen. This is an important detail, for it should be noticed that Peirce does not say that in mathematics we deal with any *groups* or *sets* of objects or entities, of whatsoever kind. In fact, the domain of the Peircean mathematical ontology may very well go beyond sets, for mathematical objects do not exist, being merely imaginary. Therefore, there is no reason why to be constrained to sets or to relations between sets. One reason for that lies in the fact that, for Peirce, the very diagram observed in mathematical reasoning is a *particular* figure itself, say, a triangle, or a square etc., which is used to stand for *any* one of a class of triangles, squares etc. that have the same form.

It is also remarkable that Peirce seems to use a highly abstract vocabulary, much more than Whitehead (at least in the writings we have been quoting from). Nevertheless, it seems both thinkers agree that mathematics *qua* mathematics has nothing to do with matters of effective existence, with matters of actuality. As Whitehead says, "we can have no *a priori* certainty we are right in believing that the observed entities in the concrete universe form a particular instance of what falls under our general reasoning." [39, p. 33]. To attain some certainty, Peirce would say, requires the whole process of inquiry, and, for that reason, any certainty will only be probable and provisional.

Maybe, the difference concerning the mathematical ontology of each thinker could lead to very different metaphysical commitments, but that is a discussion we cannot even guess upon now at all. For, regarding the issue of how to give content to abstract forms and relations, this can only be answered with a minute study that it is impossible to pursue in this work. We hope, though, that the questions here presented be developed in other inquiries by other inquirers.

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Cassiano Terra Rodrigues
Departamento de Filosofia
Pontifícia Universidade Católica de São Paulo
Rua Ministro Godoi, 969
4 andar, Bloco A, Sala 4E-16, Perdizes
CEP 05015-901 São Paulo, SP, Brazil
e-mail: ctrodrigues@pucsp.br

Received: September 27, 2010.
Accepted: February 1, 2011.